

Multigraded Betti numbers of some path ideals

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Abstract

We determine (multi)graded Betti numbers of path ideals of lines and star graphs.

1 Introduction

The path ideal of a directed graph was introduced by Conca and De Negri [7] and recently these ideals have been studied by many authors, see [1, 2, 3, 5, 6, 9, 11, 12]. In this paper we consider the path ideals of undirected graphs. In particular, we give formulas for Betti numbers of path ideals of lines and stars extending the work of [1].

2 Preliminaries

2.1 Simplicial complexes and homology

An **abstract simplicial complex** Δ on a set of **vertices** $V(\Delta) = \{v_1, \dots, v_n\}$ is a collection of subsets of V such that $\{v_i\} \in \Delta$ for all i and, $F \in \Delta$ implies that all subsets of F are also in Δ . The elements of Δ are called **faces** and the maximal faces under inclusion are called **facets**. If the facets F_1, \dots, F_q generate Δ , we write $\Delta = \langle F_1, \dots, F_q \rangle$ or $\text{Facets}(\Delta) = \{F_1, \dots, F_q\}$.

A face $\{v_1, v_2, \dots, v_n\} - \{v_{i_1}, \dots, v_{i_s}\}$ will be denoted by $\{v_1, \dots, \widehat{v}_{i_1}, \dots, \widehat{v}_{i_s}, \dots, v_n\}$ for $i_1 < i_2 < \dots < i_s$.

Two simplicial complexes Δ and Γ are **isomorphic** if there is a bijection $\varphi : V(\Delta) \rightarrow V(\Gamma)$ between their vertex sets such that F is a face of Δ iff $\varphi(F)$ is a face of Γ .

Let Δ and Γ be simplicial complexes which has no common vertices. Then the **join** of Δ and Γ is the simplicial complex given by

$$\Delta * \Gamma = \{\delta \cup \gamma : \delta \in \Delta, \gamma \in \Gamma\}.$$

A **cone** with **apex** v is a special join obtained by joining a simplicial complex Δ with $\{\emptyset, v\}$ where v is an element which is not in the vertex set of Δ . Equivalently, a simplicial complex is a cone with apex v if v is a member of every facet.

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For each integer i , the \mathbb{k} -vector space $\tilde{H}_i(\Delta, \mathbb{k})$ is the i^{th} **reduced homology** of Δ over \mathbb{k} . For the sake of simplicity, we drop \mathbb{k} and write $\tilde{H}_i(\Delta)$ whenever we work on a fixed ground field \mathbb{k} .

A **simplex** is a simplicial complex that contains all subsets of its nonempty vertex set. The **boundary** Σ of a simplex $\Delta = \langle \{v_1, \dots, v_n\} \rangle$ is obtained from Δ by removing the maximal face of Δ . And, the homology groups of Σ are given by

$$\tilde{H}_p(\Sigma, \mathbb{k}) \cong \begin{cases} \mathbb{k} & \text{if } p = n - 2 \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

The **irrelevant complex** $\{\emptyset\}$ has the homology groups

$$\tilde{H}_p(\{\emptyset\}, \mathbb{k}) \cong \begin{cases} \mathbb{k} & \text{if } p = -1 \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

whereas the **void complex** $\{\}$ has trivial reduced homology in all degrees.

A simplicial complex Δ is **acyclic** (over \mathbb{k}) if $\tilde{H}_i(\Delta, \mathbb{k})$ is trivial for all i . Examples of acyclic complexes include cones and simplices.

The homology of two simplicial complexes is related to homology of their union and intersection by the Mayer-Vietoris long exact sequence:

Theorem 2.1 (Corollary 6.4, [13]). *Let Δ_1 and Δ_2 be two simplicial complexes. Then there is a long exact sequence*

$$\cdots \rightarrow \tilde{H}_p(\Delta_1) \oplus \tilde{H}_p(\Delta_2) \rightarrow \tilde{H}_p(\Delta_1 \cup \Delta_2) \rightarrow \tilde{H}_{p-1}(\Delta_1 \cap \Delta_2) \rightarrow \tilde{H}_{p-1}(\Delta_1) \oplus \tilde{H}_{p-1}(\Delta_2) \rightarrow \cdots \quad (3)$$

where the homology can be taken over any field.

A particular case of Theorem 2.1 occurs when a simplicial complex $\Delta = \Delta_1 \cup \Delta_2$ is a union of two acyclic subcomplexes Δ_1 and Δ_2 . In that case, the sequence (3) becomes

$$\cdots \rightarrow 0 \rightarrow \tilde{H}_p(\Delta_1 \cup \Delta_2) \rightarrow \tilde{H}_{p-1}(\Delta_1 \cap \Delta_2) \rightarrow 0 \rightarrow \cdots$$

whence $\tilde{H}_p(\Delta_1 \cup \Delta_2)$ and $\tilde{H}_{p-1}(\Delta_1 \cap \Delta_2)$ are isomorphic for all p . Since we will make frequent use of this specific case we state it separately as an immediate Corollary.

Corollary 2.2. *If Δ_1 and Δ_2 are acyclic simplicial complexes then*

$$\tilde{H}_p(\Delta_1 \cup \Delta_2) \cong \tilde{H}_{p-1}(\Delta_1 \cap \Delta_2)$$

for every p and the homology can be taken over any field.

2.2 Graphs and resolutions

Let $S = \mathbb{k}[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field \mathbb{k} . Given a minimal multigraded free resolution

$$0 \longrightarrow \bigoplus_{\mathbf{m} \in \mathbb{N}^n} S(-\mathbf{m})^{b_{r,\mathbf{m}}(I)} \xrightarrow{\partial_r} \dots \longrightarrow \bigoplus_{\mathbf{m} \in \mathbb{N}^n} S(-\mathbf{m})^{b_{1,\mathbf{m}}(I)} \xrightarrow{\partial_1} \bigoplus_{\mathbf{m} \in \mathbb{N}^n} S(-\mathbf{m})^{b_{0,\mathbf{m}}(I)} \xrightarrow{\partial_0} I \longrightarrow 0$$

of I , the associated ranks $b_{i,\mathbf{m}}(I)$ are called **multigraded Betti numbers** of I . Graded and multigraded Betti numbers are related by the equation

$$b_{i,j}(I) = \sum_{\deg(\mathbf{m})=j} b_{i,\mathbf{m}}(I) \quad (4)$$

where $\deg(\mathbf{m})$ stands for the standard degree of \mathbf{m} (i.e $\deg(x_1^{a_1} \dots x_n^{a_n}) = a_1 + \dots + a_n$).

For a **graph** G , the vertex and edge sets are denoted by $V(G)$ and $E(G)$ respectively. All graphs in this paper should be assumed simple meaning that loopless and without multiple edges. Two vertices u and v are **adjacent** to one another if $\{u, v\}$ is an edge of G . A vertex u of G is called an **isolated vertex** if it is not adjacent to any vertex of G . We will say that G is of **size** e and of **order** n if it has e edges and n vertices. For two vertices u and v of G , a **path** of length $t - 1$ from u to v is a sequence of $t \geq 2$ distinct vertices $u = z_1, \dots, z_t = v$ such that $\{z_i, z_{i+1}\} \in E(G)$ for all $i = 1, \dots, t - 1$. We will denote by L_n a line of order n . Also C_n and \mathcal{S}_n will be a cyle and a star graph of size n respectively.

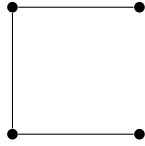


Figure 1: L_4

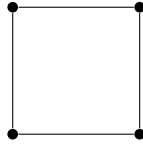


Figure 2: C_4

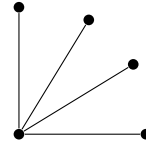
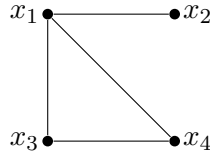


Figure 3: \mathcal{S}_4

If G is a graph with vertex set $V = \{x_1, \dots, x_n\}$ then its **path ideal** $I_t(G)$ is the monomial ideal of $S = \mathbb{k}[x_1, \dots, x_n]$ given by

$$I_t(G) = (x_{i_1} \dots x_{i_t} \mid x_{i_1}, \dots, x_{i_t} \text{ is a path of length } t - 1 \text{ in } G).$$

Example 2.3. A graph G of order 4 which has the path ideals $I_4(G) = (x_2 x_1 x_4 x_3)$, $I_3(G) = (x_2 x_1 x_4, x_2 x_1 x_3, x_1 x_3 x_4)$, $I_2(G) = (x_1 x_2, x_1 x_3, x_1 x_4, x_3 x_4)$, $I_1(G) = (x_1, x_2, x_3, x_4)$.



For a square-free monomial m we denote by G_m the **induced subgraph** of G on the set of vertices that divide m .

Let $I = (m_1, \dots, m_s)$ be a monomial ideal of S which is minimally generated by the set of monomials $M = \{m_1, \dots, m_s\}$. The **Taylor simplex** Θ of I is a simplex on s vertices which are labelled with the minimal generators of I . If $\tau = \{m_{i_1}, \dots, m_{i_r}\}$ is a face of Θ , then by $\text{lcm}(\tau)$ we mean $\text{lcm}(m_{i_1}, \dots, m_{i_r})$. For any monomial m in S ,

$$\Theta_{\leq m} = \{\tau \in \Theta \mid \text{lcm}(\tau) \text{ divides } m\}$$

and

$$\Theta_{< m} = \{\tau \in \Theta \mid \text{lcm}(\tau) \text{ strictly divides } m\}$$

are subcomplexes of Θ . Clearly we have the equation

$$\Theta_{< m} = \bigcup_{x_i \mid m} \Theta_{\leq \frac{m}{x_i}}$$

and every facet of $\Theta_{\leq \frac{m}{x_i}}$ is of the form

$$F_i := V(\Theta_{\leq m}) - \{u \in M \mid x_i \text{ does not divide } u\}.$$

Therefore we have

$$F_i \in \text{Facets}(\Theta_{< m}) \Leftrightarrow F_i \text{ is maximal in } \{F_j \mid x_j \text{ divides } m\}. \quad (5)$$

The following Theorem will be our main tool to calculate Betti numbers in this paper.

Theorem 2.4 ([4]). *Let I be a monomial ideal of S which is minimally generated by the monomials m_1, \dots, m_s . Denote by Θ the Taylor simplex of I . For $i \geq 1$, the multigraded Betti numbers of S/I are given by*

$$b_{i,m}(S/I) = \begin{cases} \dim_{\mathbb{k}} \tilde{H}_{i-2}(\Theta_{< m}; \mathbb{k}) & \text{if } m \text{ divides } \text{lcm}(m_1, \dots, m_s) \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

Remark 2.5. If $I = (m_1, \dots, m_s)$ and $q = \deg \text{lcm}(m_1, \dots, m_s)$ then for any $r > q$ we have $b_{i,r}(I) = 0$ for all i . Therefore we call the numbers $b_{i,q}(I), i \in \mathbb{Z}$ as the **top grade Betti numbers**.

Remark 2.6. Suppose that Δ is the Taylor simplex of $I(G)$ for some graph G . If the induced graph G_m contains an isolated vertex, then $\Delta_{< m} = \Delta$ is a simplex. So $b_{i,m}(S/I(G)) = 0$ for all i by Theorem 2.4.

Lemma 2.7 ([8]). *If I_1, I_2, \dots, I_N are square-free monomial ideals whose minimal generators contain no common variable and each I_i has minimal generators whose least common multiple is of degree q_i , then*

$$b_{i, q_1 + \dots + q_N}(S/(I_1 + I_2 + \dots + I_N)) = \sum_{u_1 + \dots + u_N = i} b_{u_1, q_1}(S/I_1) \dots b_{u_N, q_N}(S/I_N). \quad (7)$$

Lemma 2.8. *If m is a square-free monomial of degree j and $t \geq 2$, then $b_{i,m}(S/I_t(G)) = b_{i,j}(S/I_t(G_m))$.*

Proof. Proof is similar to Lemma 3.1 in [8]. □

3 Betti numbers of some path ideals

Definition 3.1. For any $n \geq t \geq 1$ the simplicial complex Ω_t^n on the set of vertices $\{1, \dots, n\}$ is defined by

$$\text{Facets}(\Omega_t^n) = \left\{ \{1, \dots, \hat{i}, \widehat{i+1}, \dots, \widehat{i+t-1}, i+t, \dots, n\} \mid i = 1, \dots, n-t+1 \right\}.$$

Example 3.2. For $n = 5$ and $t = 2$ the simplicial complex Ω_2^5 has facets $\{\hat{1}, \hat{2}, 3, 4, 5\}$, $\{1, \hat{2}, \hat{3}, 4, 5\}$, $\{1, 2, \hat{3}, \hat{4}, 5\}$ and, $\{1, 2, 3, \hat{4}, \hat{5}\}$.

Remark 3.3. For $n = t$ the simplicial complex Ω_t^n is the irrelevant complex $\{\emptyset\}$. If $t = 1$ then Ω_1^n coincides with the boundary of an $n - 1$ dimensional simplex.

As the simplicial complex Ω_t^n will come up in the next sections, we study its homology groups.

Lemma 3.4. For $n \geq 2t + 1$ we have $\tilde{H}_p(\Omega_t^n) \cong \tilde{H}_{p-2}(\Omega_t^{n-t-1})$. Otherwise,

$$\tilde{H}_p(\Omega_t^n) \cong \begin{cases} \tilde{H}_p(\{\emptyset\}) & \text{if } n = t \\ \tilde{H}_{p-1}(\{\emptyset\}) & \text{if } n = t + 1 \\ 0 & \text{if } t + 2 \leq n \leq 2t. \end{cases} \quad (8)$$

Proof. The case $n = t$ is clear as $\Omega_t^t = \{\emptyset\}$. So we assume that $n > t$ and fix an index p . We write $\Omega_t^n = S \cup C$ where S is the simplex on vertices $\{t+1, \dots, n\}$ and C is the cone generated by the facets of Ω_t^n that contain the vertex 1. Note that by Corollary 2.2 we have

$$\tilde{H}_p(\Omega_t^n) \cong \tilde{H}_{p-1}(S \cap C).$$

We consider the three cases left:

Case 1: If $n = t + 1$ then $S \cap C$ is the irrelevant complex and we are done.

Case 2: If $t + 2 \leq n \leq 2t$ then $S \cap C$ is a simplex whose maximal face is $\{t+2, \dots, n\}$.

Case 3: If $n \geq 2t + 1$ then it is not hard to check that $S \cap C$ can be written as a union $S \cap C = S_1 \cup C_1$ where $S_1 = \langle \{t+2, \dots, n\} \rangle$ and, C_1 is the cone with apex $t+1$ such that

$$\text{Facets}(C_1) = \left\{ \{t+1, \dots, n\} - \{i, i+1, \dots, i+t-1\} \mid i = t+2, \dots, n-t+1 \right\}.$$

Now observe that $S_1 \cap C_1 \cong \Omega_t^{n-t-1}$ and again by Corollary 2.2 we get $\tilde{H}_{p-1}(S \cap C) \cong \tilde{H}_{p-2}(S_1 \cap C_1) \cong \tilde{H}_{p-2}(\Omega_t^{n-t-1})$ which completes the proof. \square

Theorem 3.5. The homology groups of Ω_t^n are given by

$$\tilde{H}_p(\Omega_t^n) \cong \begin{cases} \tilde{H}_{p+1-\frac{2n}{t+1}}(\{\emptyset\}) & \text{if } n \equiv 0 \pmod{t+1} \\ \tilde{H}_{p+2-\frac{2(n+1)}{t+1}}(\{\emptyset\}) & \text{if } n \equiv t \pmod{t+1} \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

Proof. Follows by a straightforward induction using Lemma 3.4. \square

Corollary 3.6. *The dimensions of reduced homologies of Ω_t^n are independent of the ground field. And they are given by*

$$\dim \tilde{H}_p(\Omega_t^n) = \begin{cases} \delta_{p+2, \frac{2n}{t+1}} & \text{if } n \equiv 0 \pmod{t+1} \\ \delta_{p+3, \frac{2(n+1)}{t+1}} & \text{if } n \equiv t \pmod{t+1} \\ 0 & \text{otherwise.} \end{cases} \quad (10)$$

Proof. Follows by Theorem 3.5 and Equation (2). \square

3.1 Lines and Cycles

Throughout this section let Δ be the Taylor simplex of $I_t(L_n)$ where L_n is a line on vertices x_1, \dots, x_n . If $n < t$ then there is no path on L_n of length $t-1$, and therefore $I_t(L_n) = 0$. Let us assume $n \geq t$ then we have

$$\Delta = \langle x_i x_{i+1} \dots x_{i+t-1} \mid i = 1, \dots, n-t+1 \rangle.$$

For simplicity, we replace the label of a vertex $x_i x_{i+1} \dots x_{i+t-1}$ with i for all $i = 1, \dots, n-t+1$. Hence Δ can be viewed as a simplex with maximal face $\{1, 2, \dots, n-t+1\}$. Now we want to find $\Delta_{<m}$. Following the Equation (5), the maximal elements of

$$\begin{aligned} & \{\{\hat{1}, 2, \dots, n-t+1\}, \{1, \dots, n-t, n-\hat{t}+1\}\} \\ & \cup \{\{\hat{1}, \dots, \hat{i}, i+1, \dots, n-t+1\} \mid i = 2, \dots, t\} \\ & \cup \{\{1, \dots, i-1, \hat{i}, \widehat{i+1}, \dots, \widehat{i+t-1}, i+t, \dots, n-t+1\} \mid i = 2, \dots, n-2t+2\} \\ & \cup \{\{1, \dots, i-1, \hat{i}, \dots, \widehat{n-t+1}\} \mid i = n-2t+3, \dots, n-t\} \end{aligned}$$

give the facets of $\Delta_{<m}$. Therefore, if $n < 2t+1$ then

$$\Delta_{<m} = \langle \{\hat{1}, 2, \dots, n-t+1\}, \{1, \dots, n-t, n-\widehat{t}+1\} \rangle. \quad (11)$$

And, if $n \geq 2t+1$ we have the following equation.

$$\begin{aligned} \text{Facets}(\Delta_{<m}) = & \{\{\hat{1}, 2, \dots, n-t+1\}, \{1, \dots, n-t, n-\widehat{t}+1\}\} \\ & \cup \{\{1, \dots, i-1, \hat{i}, \widehat{i+1}, \dots, \widehat{i+t-1}, i+t, \dots, n-t+1\} \mid i = 2, \dots, n-2t+1\} \end{aligned} \quad (12)$$

Theorem 3.7 (Top grade Betti numbers of path ideals of lines). *For all $i \geq 1$, and $n \geq 1$, we have*

$$b_{i,n}(S/I_t(L_n)) = \begin{cases} \delta_{i, \frac{2n}{t+1}} & \text{if } n \equiv 0 \pmod{t+1} \\ \delta_{i+1, \frac{2n+2}{t+1}} & \text{if } n \equiv t \pmod{t+1} \\ 0 & \text{otherwise.} \end{cases} \quad (13)$$

Proof. First suppose that $n < t$, then we know that $I_t(L_n) = 0$. As n cannot be 0 or $t \bmod t + 1$ in this case we are done.

Now we assume that $n \geq t$. Let m be the product of vertices of L_n . By Equation (4) and Theorem 2.4 we have

$$b_{i,n}(S/I_t(L_n)) = b_{i,m}(S/I_t(L_n)) = \dim_{\mathbb{k}} \tilde{H}_{i-2}(\Delta_{<m}, \mathbb{k}).$$

We consider two cases:

Case 1: Suppose that $n < 2t + 1$. By Equation (11) we have

$$\Delta_{<m} = \langle \hat{1}, 2, \dots, n - t + 1 \rangle, \langle 1, \dots, n - t, \widehat{n - t + 1} \rangle.$$

Then we have three cases to prove. If $n = t$, then $\Delta_{<m} = \{\emptyset\}$ and so that

$$\dim_{\mathbb{k}} \tilde{H}_{i-2}(\Delta_{<m}, \mathbb{k}) = \dim_{\mathbb{k}} \tilde{H}_{i-2}(\{\emptyset\}, \mathbb{k}) = \delta_{i-2, -1}.$$

Observe that $\delta_{i-2, -1} = \delta_{i+1, \frac{2n+2}{t+1}}$ for $n = t$ which proves Equation (13) for this case. Now observe that if $n > t$ then $\Delta_{<m}$ is a union of two simplices

$$\Delta_{<m} = \langle \hat{1}, 2, \dots, n - t + 1 \rangle \cup \langle 1, \dots, n - t, \widehat{n - t + 1} \rangle = S_1 \cup S_2.$$

Hence by Corollary 2.2, $\dim_{\mathbb{k}} \tilde{H}_{i-2}(\Delta_{<m}, \mathbb{k}) \cong \dim_{\mathbb{k}} \tilde{H}_{i-3}(S_1 \cap S_2, \mathbb{k})$. If $n = t + 1$ then $S_1 \cap S_2$ is the irrelevant complex. Therefore we have

$$\dim_{\mathbb{k}} \tilde{H}_{i-3}(S_1 \cap S_2, \mathbb{k}) \cong \dim_{\mathbb{k}} \tilde{H}_{i-3}(\{\emptyset\}, \mathbb{k}) = \delta_{i-3, -1}.$$

Now we check that indeed $\delta_{i-3, -1} = \delta_{i, \frac{2n}{t+1}}$ for $n = t + 1$ and the proof follows for this case. Next, if $n \geq t + 1$ then $S_1 \cap S_2$ is a simplex and has trivial reduced homology in all degrees.

Case 2: Suppose that $n \geq 2t + 1$. Then by Equation (12) we have $\Delta_{<m} = S_1 \cup S_2 \cup \Upsilon$ where $\Upsilon = \langle \{1, \dots, i - 1, \hat{i}, \widehat{i + 1}, \dots, \widehat{i + t - 1}, i + t, \dots, n - t + 1\} \mid i = 2, \dots, n - 2t + 1 \rangle$, $S_1 = \langle \hat{1}, 2, \dots, n - t + 1 \rangle$ and $S_2 = \langle \{1, \dots, n - t, \widehat{n - t + 1}\} \rangle$. Now we write $\Delta_{<m}$ as a union $\Delta_{<m} = S_1 \cup (S_2 \cup \Upsilon)$ where $S_2 \cup \Upsilon$ is a cone with apex 1. By virtue of Corollary 2.2 we have

$$\dim_{\mathbb{k}} \tilde{H}_{i-2}(\Delta_{<m}, \mathbb{k}) \cong \dim_{\mathbb{k}} \tilde{H}_{i-3}(S_1 \cap (S_2 \cup \Upsilon), \mathbb{k}).$$

Now observe that $S_1 \cap (S_2 \cup \Upsilon) = C \cup S_2$ where C is the cone generated by the facets of $S_1 \cap (S_2 \cup \Upsilon)$ that contain the vertex $n - t + 1$. Again by Corollary 2.2 we get

$$\dim_{\mathbb{k}} \tilde{H}_{i-3}(S_1 \cap (S_2 \cup \Upsilon), \mathbb{k}) \cong \dim_{\mathbb{k}} \tilde{H}_{i-4}(C \cap S_2, \mathbb{k}).$$

Note that $C \cap S_2$ is isomorphic to the simplicial complex Ω_t^{n-t-1} and by Corollary 3.6 we have.

$$\dim \tilde{H}_{i-4}(\Omega_t^{n-t-1}) = \begin{cases} \delta_{i-2, \frac{2(n-t-1)}{t+1}} & \text{if } n \equiv 0 \bmod t + 1 \\ \delta_{i-1, \frac{2(n-t)}{t+1}} & \text{if } n \equiv t \bmod t + 1 \\ 0 & \text{otherwise} \end{cases} \quad (14)$$

which agrees with the formula given in Equation (13). \square

Theorem 3.8 (Multigraded Betti numbers of path ideals of lines). *Let $t \geq 2$ and m be a squarefree monomial of degree j . Then the multigraded Betti number $b_{i,m}(S/I_t(L_n)) = 1$ if the induced graph $(L_n)_m$ consists of a collection of disjoint lines that satisfy the following conditions:*

- (i) *Each line is of order 0 or $t \bmod t+1$*
- (ii) *The number of lines of order $t \bmod t+1$ is equal to $\frac{i(t+1)-2j}{1-t}$.*

Otherwise, $b_{i,m}(S/I_t(L_n)) = 0$.

Proof. Let $(L_n)_m = \cup_{l=1}^p Q_l$ be a disjoint union of lines where each Q_l is a line of order v_l . We have

$$\begin{aligned} b_{i,m}(S/I_t(L_n)) &= b_{i,j}(S/I_t((L_n)_m)) \text{ by Lemma 2.8} \\ &= \sum_{u_1+\dots+u_p=i} b_{u_1,v_1}(S/I_t(Q_1)) \dots b_{u_p,v_p}(S/I_t(Q_p)) \text{ by Equation (7).} \end{aligned}$$

By Theorem 3.7 if one of Q_l is not of order 0 or $t \bmod t+1$ then the sum above is zero. So without loss of generality let us assume that Q_1, \dots, Q_z are of order $0 \bmod t+1$ and Q_{z+1}, \dots, Q_p are of order $t \bmod t+1$ for some $0 \leq z \leq p$. Again by Theorem 3.7, the sum above is equal to 1 if

$$\sum_{l=1}^z \frac{2v_l}{t+1} + \sum_{l=z+1}^p \left(\frac{2v_l+2}{t+1} - 1 \right) = i \quad (15)$$

and zero otherwise. Observe that (15) holds iff $p - z = \frac{i(t+1)-2j}{1-t}$ since $v_1 + \dots + v_p = j$. Hence the result follows. \square

Corollary 3.9. *If L is a line, $b_{i,j}(S/I_t(L))$ is the number of ways of choosing a collection of disjoint induced lines of L that satisfy the following conditions:*

- (i) *The orders of the lines add up to j*
- (ii) *Each line is of order 0 or $t \bmod t+1$*
- (iii) *The number of lines of order $t \bmod t+1$ is equal to $\frac{i(t+1)-2j}{1-t}$.*

Proof. Immediately follows by Theorem 3.8 and Equation (4). \square

Using the multigraded Betti numbers, we can calculate graded Betti numbers. The following result was also proved in [2].

Theorem 3.10 (Graded Betti numbers of path ideals of lines). *For $t \geq 2$, the nonzero graded Betti numbers of $S/I_t(L_n)$ are given by*

$$b_{i,j}(S/I_t(L_n)) = \binom{n-j+1}{\frac{i(t+1)-2j}{1-t}} \binom{n-j+\frac{j-ti}{1-t}}{n-j}$$

provided that n, i and j satisfy the following relations.

- (i) $n \geq j$
- (ii) $j \geq t \left(\frac{i(t+1)-2j}{1-t} \right) \geq 0$
- (iii) $n - j \geq \frac{i(t+1)-2j}{1-t} - 1$

Otherwise, the graded Betti numbers are zero.

Proof. By Lemma 3.9 it is clear that $b_{i,j}(S/I_t(L_n)) = 0$ if the condition (i) or (ii) fails. So let us assume that (i) and (ii) hold.

Now suppose that we have chosen a collection of disjoint induced lines Q_1, \dots, Q_p of L_n as in Lemma 3.9. Since the orders of Q_1, \dots, Q_p add up to j , we have $j = |\cup_{k=1}^p V(Q_k)|$. Also as the number of lines of order $t \bmod t+1$ is equal to $\frac{i(t+1)-2j}{1-t}$, at least $t \left(\frac{i(t+1)-2j}{1-t} \right)$ vertices of $\cup_{k=1}^p V(Q_k)$ belong to a line of order $t \bmod t+1$. Therefore at most $j - t \left(\frac{i(t+1)-2j}{1-t} \right) = (1+t) \left(\frac{j-ti}{1-t} \right)$ vertices of $\cup_{k=1}^p V(Q_k)$ belong to a line of order $0 \bmod t+1$. Now it becomes clear that the problem of choosing a collection of disjoint induced lines of L_n that is described in Lemma 3.9 corresponds to the problem of ordering $\frac{i(t+1)-2j}{1-t}$ many “ t ”s, $\frac{j-ti}{1-t}$ many “ $1+t$ ”s and “ $n-j$ ” many points on a row such that there is a point between any “ t ”s and the order of “ t ”s and “ $t+1$ ”s between two points is ignored. (Note that for example, in the latter interpretation the orderings

$$\cdot t (t+1) (t+1) \cdot \cdot (t+1) t \cdot t \text{ and } \cdot (t+1) t (t+1) \cdot \cdot t (t+1) \cdot t$$

are considered as the same since they both correspond to the collection $L_{t+2(t+1)}, L_{(t+1)+t}, L_t$ where

$$L_n = L_{3(t+1)+3t+4} = L_1 \cup L_{t+2(t+1)} \cup L_1 \cup L_1 \cup L_{(t+1)+t} \cup L_1 \cup L_t$$

Now to count the number of solutions to this problem we spread $\frac{i(t+1)-2j}{1-t}$ many “ t ”s on a row and put one point between any two:

$$t \cdot t \cdot t \cdot \dots \cdot t$$

Observe that to achieve this there must be at least $\frac{i(t+1)-2j}{1-t} - 1$ many points, i.e. $n - j \geq \frac{i(t+1)-2j}{1-t} - 1$ which is condition (iii).

Now we are allowed to insert the remaining $n - j - \left(\frac{i(t+1)-2j}{1-t} - 1 \right)$ points. Observe that we have $\frac{i(t+1)-2j}{1-t} + 1$ many places to put each of them as indicated with $-$ below.

$$- t - \cdot t - \cdot t - \cdot \dots - \cdot t -$$

This is equivalent to finding the number of integer solutions to the equation

$$A_1 + A_2 + \dots + A_{\frac{i(t+1)-2j}{1-t}+1} = n - j - \left(\frac{i(t+1)-2j}{1-t} - 1 \right)$$

with $A_i \geq 0$, which is $\binom{n-j+1}{\frac{i(t+1)-2j}{1-t}}$.

Finally we insert the “ $t + 1$ ”s. Since the order of “ t ”s and “ $t + 1$ ”s between two points is ignored there are $n - j + 1$ places (spaces between two points plus endpoints) to insert each $t + 1$. But the number of ways of doing this is equal to the number of integer solutions of the equation

$$A_1 + A_2 + \dots + A_{n-j+1} = \frac{j - ti}{1 - t}$$

with $A_i \geq 0$, which is $\binom{n-j+\frac{j-ti}{1-t}}{n-j}$. Hence the number of all possible collections is equal to

$$\binom{n-j+1}{\frac{i(t+1)-2j}{1-t}} \binom{n-j+\frac{j-ti}{1-t}}{n-j}$$

and the proof is completed. \square

Corollary 3.11 (Multigraded Betti numbers of path ideals of cycles). *Let $t \geq 2$ and m be a squarefree monomial of degree $j < n$. Then the multigraded Betti number $b_{i,m}(S/I_t(C_n)) = 1$ if the induced graph $(C_n)_m$ consists of a collection of disjoint lines that satisfy the following conditions:*

- (i) *Each line is of order 0 or $t \bmod t + 1$*
- (ii) *The number of lines of order $t \bmod t + 1$ is equal to $\frac{i(t+1)-2j}{1-t}$.*

Otherwise, $b_{i,m}(S/I_t(L_n)) = 0$.

Proof. By Lemma 2.8 we have $b_{i,m}(S/I_t(C_n)) = b_{i,j}(S/I_t((C_n)_m))$. Since $(C_n)_m$ is a disjoint union of lines the proof follows by Theorem 3.8. \square

In the next Corollary, we will give a formula for the graded Betti numbers $b_{i,j}(S/I_t(C_n))$ when $j < n$. Note that Theorem 4.13 of [1] gives a complete formula for all Betti numbers.

Corollary 3.12 (Graded Betti numbers of path ideals of cycles). *For $j < n$ and $t \geq 2$ the graded Betti numbers of $S/I_t(C_n)$ are given by*

$$b_{i,j}(S/I_t(C_n)) = \frac{n}{n-j} \binom{n-j}{\frac{i(t+1)-2j}{1-t}} \binom{n-j-1+\frac{j-ti}{1-t}}{n-j-1}$$

provided that

- (i) $n - 1 \geq j$
- (ii) $j \geq t \left(\frac{i(t+1)-2j}{1-t} \right) \geq 0$
- (iii) $n - j \geq \frac{i(t+1)-2j}{1-t}$.

Otherwise, the graded Betti numbers are zero.

Proof. By Corollary 3.11 and Equation (4), $b_{i,j}(S/I_t(C_n))$ is the number of ways one can choose a collection of disjoint induced lines on C_n such that the orders of the lines add up to j , each line is of order 0 or $t \bmod t+1$ and, the number of lines of order $t \bmod t+1$ is equal to $\frac{i(t+1)-2j}{1-t}$. Then this is a problem of ordering $\frac{i(t+1)-2j}{1-t}$ many “ t ”s, $\frac{j-ti}{1-t}$ many “ $t+1$ ”s and $n-j$ many points around a circle such that there is at least one point between any “ t ”s and the order of “ t ”s and “ $t+1$ ”s between two points is ignored. Any such ordering can be obtained by first fixing a point on the cycle and ordering the remaining $n-j-1$ points, $\frac{i(t+1)-2j}{1-t}$ many “ t ”s, $\frac{j-ti}{1-t}$ many “ $t+1$ ”s on a row with the same conditions. By Theorem 3.10, there are $\binom{n-j}{\frac{i(t+1)-2j}{1-t}} \binom{n-j-1+\frac{j-ti}{1-t}}{n-j-1}$ ways to do it. Also there are n choices to fix a vertex on the cycle. However it is clear that $n \binom{n-j}{\frac{i(t+1)-2j}{1-t}} \binom{n-j-1+\frac{j-ti}{1-t}}{n-j-1}$ will give an overcount since fixing different points may yield the same ordering. To overcome this problem, consider a circle with a desired ordering. It has $n-j$ points and this ordering was counted once for fixing each of these points. Hence the result follows. \square

3.2 Stars

Throughout this section \mathcal{S}_n will be a star graph of size n .

Lemma 3.13. *Let G be a connected graph, let Δ be the Taylor simplex of $I_2(G)$. If m is the product of the vertices of G then the simplicial complex $\Delta_{<m}$ is the boundary of Δ iff G is a star.*

Proof. Suppose that e_1, \dots, e_q are the edges of the graph G . Then, $\Delta_{<m}$ is the boundary of Δ iff $F_i = \{e_1, \dots, e_q\} - \{e_i\}$ is a facet of $\Delta_{<m}$ for each $i = 1, \dots, q$. The latter holds only if multidegree of F_i properly divides m for every i . Or, equivalently each e_i contains a vertex x_i such that $x_i \notin \cup_{j \neq i} e_j$. But this happens only if G is a star since G is connected. \square

Corollary 3.14. *Let G be a star on $d+1$ vertices. Then*

$$b_{i,d+1}(S/I(G)) = \begin{cases} 1 & \text{if } i = d \\ 0 & \text{otherwise} \end{cases}$$

Proof. Follows by combining Lemma 3.13, Theorem 2.4 and Equation (1). \square

Corollary 3.15. *Let G be a star on $d+1$ vertices. Then the graded Betti numbers of $I(G)$ are given by*

$$b_{i,d+1-j}(S/I(G)) = \begin{cases} \binom{d}{j} & i = d - j \\ 0 & \text{otherwise} \end{cases}$$

Proof. Fix j and recall Equation (4) and Lemma 2.8. Any induced subgraph of G is either a star or contains isolated vertex. If it contains an isolated vertex then by Remark 2.6 the multigraded Betti number for such induced subgraph is zero. Hence by Corollary 3.14 we see that $b_{i,d+1-j}(S/I(G))$ is the number of induced star subgraphs of G of order $d+1-j$ if $i = d-j$ and zero otherwise. \square

Proposition 3.16. *Let Γ be a simplicial complex which is not a cone. Suppose that $\langle F_1, \dots, F_q \rangle = \Gamma$ and there exists a sequence of distinct vertices v_1, \dots, v_q of Γ such that $v_i \notin F_j$ iff $i = j$. Then $\tilde{H}_p(\Gamma, \mathbb{k}) \cong \tilde{H}_{p-q+1}(\{\emptyset\}, \mathbb{k})$ for any field \mathbb{k} .*

Proof. We induct on q , the number of facets. Since there is no simplex which satisfies the assumptions of the given Proposition, the basis step starts at $q = 2$.

Suppose that $\Gamma = \langle F_1, F_2 \rangle$ is not a cone and it has two vertices v_1, v_2 such that $v_i \notin F_j \Leftrightarrow i = j$. Then $\langle F_1 \rangle \cap \langle F_2 \rangle \cong \{\emptyset\}$. Since $\Gamma = \langle F_1 \rangle \cup \langle F_2 \rangle$ and $\langle F_1 \rangle, \langle F_2 \rangle$ are acyclic, by virtue of Corollary 2.2 we have $\tilde{H}_p(\Gamma) \cong \tilde{H}_{p-1}(\langle F_1 \rangle \cap \langle F_2 \rangle) = \tilde{H}_{p-1}(\{\emptyset\})$ as desired.

Now let $\Gamma = \langle F_1, \dots, F_q \rangle, q \geq 3$ be a simplicial complex as in the statement of the Proposition. We write

$$\Gamma = \langle F_1, \dots, F_{q-1} \rangle \cup \langle F_q \rangle$$

where $\langle F_q \rangle$ is a simplex and $\langle F_1, \dots, F_{q-1} \rangle$ is a cone with apex v_q . By Corollary 2.2 we have $\tilde{H}_p(\Gamma) \cong \tilde{H}_{p-1}(\langle F_1, \dots, F_{q-1} \rangle \cap \langle F_q \rangle)$. But observe that

$$\langle F_1, \dots, F_{q-1} \rangle \cap \langle F_q \rangle = \langle F_1 \cap F_q, \dots, F_{q-1} \cap F_q \rangle$$

as $v_j \in F_i \cap F_q, v_j \notin F_j \cap F_q$ so that $F_i \cap F_q \not\subseteq F_j \cap F_q$ for all $1 \leq i \neq j \leq q-1$. Clearly, the simplicial complex $\langle F_1 \cap F_q, \dots, F_{q-1} \cap F_q \rangle$ is not a cone and moreover

$$v_i \notin F_j \cap F_q \Leftrightarrow i = j$$

for all $1 \leq i, j \leq q-1$. Hence it satisfies the inductive hypothesis and we get

$$\tilde{H}_{p-1}(\langle F_1 \cap F_q, \dots, F_{q-1} \cap F_q \rangle) \cong \tilde{H}_{p-q+1}(\{\emptyset\})$$

which completes the proof. \square

Theorem 3.17. *Let \mathcal{S}_n be a star graph of size $n \geq 2$. Then for all $i \geq 1$*

$$b_{i,n+1}(S/I_3(\mathcal{S}_n)) = \begin{cases} i & \text{if } n+1 = i+2 \\ 0 & \text{otherwise.} \end{cases} \quad (16)$$

Proof. Let \mathcal{S}_n be a star graph of size n with the edge set $E(\mathcal{S}_n) = \{\{x_0, x_i\} \mid i = 1, \dots, n\}$. Suppose that $\Delta(\mathcal{S}_n)$ is the Taylor simplex of $I_3(\mathcal{S}_n)$. We prove the given statement by induction on n and using Theorem 2.4 so that for all $i \geq 1$

$$b_{i,n+1}(S/I_3(\mathcal{S}_n)) = \dim_{\mathbb{k}} \tilde{H}_{i-2}(\Delta(\mathcal{S}_n)_{<x_0 \dots x_n}, \mathbb{k}).$$

For $n = 2$, we have $\Delta(\mathcal{S}_2)_{<x_0 x_1 x_2} \cong \{\emptyset\}$ so the basis step is settled by (2).

Next we consider $\Delta(\mathcal{S}_n)_{<x_0 x_1 \dots x_n}$ for some $n \geq 3$. We have a decomposition

$$\Delta(\mathcal{S}_n)_{<x_0 x_1 \dots x_n} = \Delta(\mathcal{S}_n)_{\leq x_0 x_1 \dots x_{n-1}} \bigcup_{i=1}^{n-1} \left(\bigcup_{x_i} \Delta(\mathcal{S}_n)_{\leq \frac{x_0 x_1 \dots x_n}{x_i}} \right) \quad (17)$$

since $\Delta(\mathcal{S}_n)_{\leq x_1 \dots x_n}$ is isomorphic to the irrelevant complex. For $i \geq 1$ we set

$$\Delta(\mathcal{S}_n)_{\leq \frac{x_0 \dots x_n}{x_i}} = \langle F_i \rangle := \langle \{x_0 x_j x_k \mid j, k \in \{1, \dots, n\} - \{i\} \text{ and } j < k\} \rangle \quad (18)$$

and note that every element of $\{F_1, \dots, F_n\}$ is maximal with respect to inclusion because of the symmetry of star graphs. By Equation (5) we get $\Delta(\mathcal{S}_n)_{< x_0 \dots x_n} = \langle F_1, \dots, F_n \rangle$. Observe that (17) becomes

$$\Delta(\mathcal{S}_n)_{< x_0 \dots x_n} = \langle F_n \rangle \cup \langle F_1, \dots, F_{n-1} \rangle \quad (19)$$

by definition of F_i . Now we claim the followings:

- (i) $\langle F_n \rangle \cap \langle F_1, \dots, F_{n-1} \rangle \cong \Delta(\mathcal{S}_{n-1})_{< x_0 \dots x_{n-1}}$
- (ii) $\tilde{H}_p(\langle F_1, \dots, F_{n-1} \rangle) \cong \tilde{H}_{p-n+2}(\{\emptyset\})$.

Claim (i) is trivial as F is a facet of $\langle F_n \rangle \cap \langle F_1, \dots, F_{n-1} \rangle$ iff $F = F_n \cap F_i$ for some $1 \leq i \leq n-1$. But the latter means that F consists of all paths of the form $x_0 x_j x_k$ where $j, k \in \{x_1, \dots, x_n\} - \{x_i, x_n\}$ and $j \neq k$.

For claim (ii) we show that Proposition 3.16 applies to the simplicial complex $\langle F_1, \dots, F_{n-1} \rangle$. To this end, we first check that $\langle F_1, \dots, F_{n-1} \rangle$ is not a cone. Assume for a contradiction it is a cone with apex $x_0 x_i x_j$. Then $x_0 x_i x_j \in F_1 \cap \dots \cap F_{n-1}$ and $i, j \in \{1, \dots, n\} - \{1, \dots, n-1\}$. Thus $i = j = n$ which is a contradiction. Now let $v_1 = x_0 x_1 x_n, \dots, v_{n-1} = x_0 x_{n-1} x_n$ be a sequence of vertices of $\langle F_1, \dots, F_{n-1} \rangle$. Clearly we have $v_i \notin F_j \Leftrightarrow i = j$ which proves claim (ii). Therefore (2) yields

$$\dim \tilde{H}_p(\langle F_1, \dots, F_{n-1} \rangle) = \begin{cases} 1 & \text{if } p = n-3 \\ 0 & \text{otherwise.} \end{cases} \quad (20)$$

Also by inductive assumption and (i) we have

$$\dim \tilde{H}_p(\langle F_n \rangle \cap \langle F_1, \dots, F_{n-1} \rangle) = \begin{cases} p+2 & \text{if } p = n-4 \\ 0 & \text{otherwise.} \end{cases} \quad (21)$$

Therefore Mayer-Vietoris sequence for (19) is

$$\begin{aligned} \dots \rightarrow 0 \rightarrow 0 \rightarrow \tilde{H}_{n-1}(\Delta(\mathcal{S}_n)_{< x_0 \dots x_n}) \rightarrow 0 \rightarrow 0 \rightarrow \tilde{H}_{n-2}(\Delta(\mathcal{S}_n)_{< x_0 \dots x_n}) \rightarrow 0 \rightarrow \\ \tilde{H}_{n-3}(\langle F_1, \dots, F_{n-1} \rangle) \rightarrow \tilde{H}_{n-3}(\Delta(\mathcal{S}_n)_{< x_0 \dots x_n}) \rightarrow \tilde{H}_{n-4}(\langle F_n \rangle \cap \langle F_1, \dots, F_{n-1} \rangle) \rightarrow \\ 0 \rightarrow \tilde{H}_{n-4}(\Delta(\mathcal{S}_n)_{< x_0 \dots x_n}) \rightarrow 0 \rightarrow 0 \rightarrow \tilde{H}_{n-5}(\Delta(\mathcal{S}_n)_{< x_0 \dots x_n}) \rightarrow 0 \rightarrow 0 \rightarrow \dots \end{aligned}$$

For $i \leq n-5$ and $i \geq n-2$ we have the sequence

$$0 \rightarrow 0 \rightarrow \tilde{H}_i(\Delta(\mathcal{S}_n)_{< x_0 \dots x_n}) \rightarrow 0$$

which implies that $\tilde{H}_i(\Delta(\mathcal{S}_n)_{< x_0 \dots x_n}) = 0$. Hence Mayer-Vietoris sequence above becomes

$$\dots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \tilde{H}_{n-3}(\langle F_1, \dots, F_{n-1} \rangle) \rightarrow \tilde{H}_{n-3}(\Delta(\mathcal{S}_n)_{< x_0 \dots x_n}) \rightarrow \quad (22)$$

$$\tilde{H}_{n-4}(\langle F_n \rangle \cap \langle F_1, \dots, F_{n-1} \rangle) \rightarrow 0 \rightarrow \tilde{H}_{n-4}(\Delta(\mathcal{S}_n)_{< x_0 \dots x_n}) \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \dots \quad (23)$$

and by (23) we see that $\tilde{H}_{n-4}(\Delta(\mathcal{S}_n)_{<x_0\dots x_n}) = 0$. Hence (22) and (23) turn into

$$\dots \rightarrow 0 \rightarrow \tilde{H}_{n-3}(\langle F_1, \dots, F_{n-1} \rangle) \rightarrow \tilde{H}_{n-3}(\Delta(\mathcal{S}_n)_{<x_0\dots x_n}) \rightarrow \tilde{H}_{n-4}(\langle F_n \rangle \cap \langle F_1, \dots, F_{n-1} \rangle) \rightarrow 0 \rightarrow \dots$$

which gives that

$$\begin{aligned} \dim \tilde{H}_{n-3}(\Delta(\mathcal{S}_n)_{<x_0\dots x_n}) &= \dim \tilde{H}_{n-3}(\langle F_1, \dots, F_{n-1} \rangle) + \dim \tilde{H}_{n-4}(\langle F_n \rangle \cap \langle F_1, \dots, F_{n-1} \rangle) \\ &= 1 + (n - 2) \text{ by (20) and (21)} \\ &= n - 1 \end{aligned}$$

and, the proof is completed. \square

Theorem 3.18 (Graded Betti numbers of path ideals of stars). *Let \mathcal{S}_n be a star graph of size $n \geq 2$. For all $i \geq 1$ the nonzero graded Betti numbers of $S/I_2(\mathcal{S}_n)$ and $S/I_3(\mathcal{S}_n)$ are given by*

$$b_{i,j}(S/I_2(\mathcal{S}_n)) = \begin{cases} \binom{n}{j-1} & \text{if } i = j - 1 \\ 0 & \text{otherwise} \end{cases}$$

and,

$$b_{i,j}(S/I_3(\mathcal{S}_n)) = \begin{cases} i \binom{n}{j-1} & \text{if } i = j - 2 \\ 0 & \text{otherwise} \end{cases}$$

where $j \leq n + 1$. In particular, $S/I_t(\mathcal{S}_n)$ has a $(t - 1)$ -linear resolution.

Proof. Similar to proof of Corollary 3.15. \square

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